

Loss Tolerance with Fractal Pentagons

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Abstract A new way of addressing loss errors is introduced which combines ideas from measurement-based quantum computation and concatenated quantum codes, allowing for universal quantum computation. It is shown that for the case where leakage is detected upon measurement, the scheme performs well under 23% loss rate. For loss rates below 10% this approach performs better than the best scheme known up to date [1]. If lost qubits are tagged prior to measurement, it can tolerate up to 50% loss. The overhead per logical qubit is shown to be significantly lower than other schemes. The obtention of the threshold is entirely analytic.

Keywords Quantum Computation · Quantum Error Correction

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1 Introduction

Qubit loss is a common type of error that has to be dealt with effectively if we are to build a quantum processor. Loss is arguably easier to handle than unknown computational errors, since sometimes it can be seen merely as an error which can be located. Loss can happen as a result of fluctuations of the qubit parameters that take the state of the system out of the computational subspace, or as a result of using inefficient detectors. There exist already several works [2, 3, 4, 5, 6], each making different assumptions and taking different error models, that combine loss protection with tolerance to unknown errors and are tailored for different architectures. However the approach to loss which is closer to the spirit

of the present work was introduced by Varnava et al. [1], where they proved that universal quantum computation is possible even with a 50% loss rate provided there are no computational errors.

We combine ideas from measurement-based quantum computation [7, 8] and from the traditional approach to fault tolerance [9]. In particular, we use the five qubit code [10], which is the smallest quantum error correcting code which can correct one general Pauli error [11]. In contrast to general Pauli errors, loss errors by construction can be located, a property which is crucial in our construct.

We show that it is possible to achieve high levels of tolerance to qubit leakage without compromising the universality of the cluster state model of computation. The whole point is to simulate a noiseless measurement pattern in a noiseless cluster state, and to this aim we encode the logical qubits using the five qubit code $[5, 1, 3]_2$ concatenated with itself, in such a way that the logical operators to be measured will be spread across many physical qubits. We will see that the logical operators are defined as the tensor product of Pauli operators and only have support, that is, are different from the identity operator, on roughly $3/5$ of the total number of qubits, which allows for loss tolerance. There is a general theory for graph code concatenation [12, 13] that generalizes the present approach in the code-theoretical framework.

2 Graph States as Error Correcting Codes

To define graph states [14] it is useful to recall the mathematical definition of a graph $G = \{V, E\}$, where $V \subset \mathbb{N}$ are the vertices where the qubits sit. A graph state is uniquely defined as the common eigenstate of

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	1	2	3	4	5
K_1	X	Z	Z	X	I
K_2	I	X	Z	Z	X
K_3	X	I	X	Z	Z
K_4	Z	X	I	X	Z
\bar{X}	X	X	X	X	X
\bar{Z}	Z	Z	Z	Z	Z

Table 1 Stabilizers K_i and the logical operators of the five qubit code.

	1	2	3	4	5
K_1	X	Z	I	I	Z
K_2	Z	X	Z	I	I
K_3	I	Z	X	Z	I
K_4	I	I	Z	X	Z
K_5	Z	I	I	Z	X

 \iff

	1	2	3	4	5
K'_1	Y	Y	Z	I	Z
K'_2	Z	Y	Y	Z	I
K'_3	I	Z	Y	Y	Z
K'_4	Z	I	Z	Y	Y
\bar{X}	Z	I	I	Z	X

Table 2 Explicit transformation between the five qubit circular graph state into a version of the five qubit code.

the operators $K_i = X_i \otimes_{\{i,j\} \in E} Z_j, \forall i \in V$ where X_i and Z_i are the Pauli matrices applied to qubit i .

A constructive definition would be to initially set in each vertex in the state $|+\rangle$, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. The symbol $E \subset \mathbf{N} \times \mathbf{N}$ corresponds to edges connecting the qubits, representing entanglement in the form of a two-qubit controlled- Z operation being applied, given by $C_Z^{ij} = \text{diag}(1, 1, 1, -1)$ for qubits i and j . Indeed, it has been shown that all stabilizer codes are locally unitary equivalent to some graph state [15]. In fact graph states can be combined with non-additive classical codes to create a larger set of quantum error-correcting codes [16].

2.1 A Version of the Five Qubit Code

The five qubit code $[[5, 1, 3]]_2$ is defined by the operators in Table 1. Alternatively, one can describe the code writing its logical codewords, $|0_L\rangle$ and $|1_L\rangle$. However, using states instead of operators is rather cumbersome and abstruse, so we will describe states using their stabilizer operators. This code saturates the singleton bound [11], *i.e.* it is the smallest quantum code that protects against one Pauli error. It is not difficult to see that this is locally unitarily equivalent to a circular five qubit graph state, defined by both set of operators in Table 2.

where we have renamed the stabilizers $K'_i = K_i K_{i+1}$ and chosen \bar{X} to be any one K_i in the original description of the graph, so that $[K'_i, \bar{X}] = 0, \forall i$. Note

that $\bar{X}' \equiv \bar{X} K'_1 K'_3 = X_1 X_2 X_3 X_4 X_5$. We choose $\bar{Z} = Z_1 Z_2 Z_3 Z_4 Z_5$, such that $[K'_i, \bar{Z}] = 0, \forall i$, and the anti-commutation relation $\{\bar{X}, \bar{Z}\} = 0$ can be readily checked. It follows that the five qubit graph state in a pentagon is locally unitary equivalent to the usual five qubit code initialized in the logical $|+\rangle$ state. This construction can be seen as a special case of *Codeword Stabilized Codes* [16], in which the \bar{Z} is the word operator.

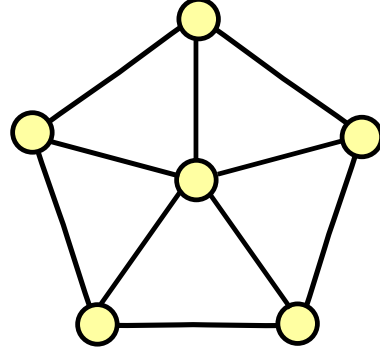


Fig. 1 Circles are qubits and lines between them represent entanglement. Five qubits in a ring (a pentagon) entangled to a centre qubit is the basic building of our scheme.

2.2 Encoding and Concatenation

Encoding a logical qubit in the five qubit graph state can be seen as a simple teleportation circuit where we have substituted one of the qubits by the five qubit graph state, as depicted in Fig. 2. Consider the state

$$|\psi\rangle_l = \frac{1}{2}(I + e_x X_l + e_y Y_l + e_z Z_l) |\psi\rangle_l, \quad (1)$$

where $e_x^2 + e_y^2 + e_z^2 = 1$ are the amplitudes containing the information. To encode this state into the graph state, the logical controlled- Z $\bar{C}_Z = C_Z^{l1} C_Z^{l2} C_Z^{l3} C_Z^{l4} C_Z^{l5}$ is applied between each of the qubits in the graph and $|\psi\rangle_l$.

Without loss of generality we only look at the X-Z ($e_y = 0$) equator of the Bloch sphere. In terms of operators, the action of the logical gate \bar{C}_Z can be described as follows:

$$\bar{C}_Z \begin{bmatrix} I_1 \\ I_5 & X_l & I_2 \\ I_4 & & I_3 \end{bmatrix} \bar{C}_Z^\dagger = \begin{bmatrix} Z_1 \\ Z_5 & X_l & Z_2 \\ Z_4 & & Z_3 \end{bmatrix}, \quad (2)$$

$$\bar{C}_Z \begin{bmatrix} Z_1 \\ X_5 & I_l & I_2 \\ Z_4 & & I_3 \end{bmatrix} \bar{C}_Z^\dagger = \begin{bmatrix} Z_1 \\ X_5 & Z_l & I_2 \\ Z_4 & & I_3 \end{bmatrix}. \quad (3)$$

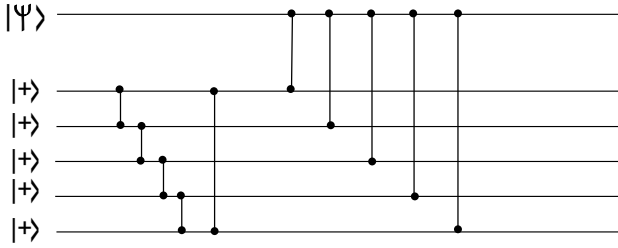


Fig. 2 The information initially contained in the centre qubit will be spread over the whole graph state after operation of the C_Z gates. Note that errors on concatenation level $N - 1$ will propagate at most to level N since $C_Z(X \otimes I)C_Z^\dagger = X \otimes Z$, $C_Z(Z \otimes I)C_Z^\dagger = Z \otimes I$

The weights e_x and e_z of X_l and Z_l will, upon measurement in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis, will be effectively stored in the amplitudes of the eigenstates of \bar{Z} and \bar{X} , respectively. The case for \bar{Y} follows through the relation $\bar{Y} = i\bar{X}\bar{Z}$. Encoding information can be seen as “expanding” the operators acting on the original qubit onto the graph state.

Now, for each qubit in the pentagon, we carry out the encoding procedure explained above in this section. This is known as *code concatenation*. To each qubit in the pentagon going out in the circuit of Fig. 2, we attach a copy of that same circuit and repeat iteratively. The logical qubit is said to be at level 0, and the qubits of the first pentagon are in level 1. Now each qubit in level 1 is encoded using the same code, which will create the concatenation level 2. Iterating this procedure amounts to concatenating the code with itself N times, in such a way that the last level of concatenation has $Q = 5^N$ physical qubits encoding one logical qubit (see Fig. 3). It is important to realize that all previous $N - 1$ levels are measured out in the $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ basis, without being exposed to loss, since they are just part of the construction procedure of the code.

For Pauli errors, the main idea behind code concatenation for a code correcting up to one error, is that the effective probability at level N decreases depends on the effective probability at level $N - 1$ as:

$$P_N^{eff} = \gamma(P_{N-1}^{eff})^2. \quad (4)$$

This can be understood as using quantum code to reduce the effective error probability of a qubit in the immediately lower level of concatenation. Since the probability of logical error will be the probability of two errors happening times some combinatorial constant γ

which is particular to each code and error model. This gives:

$$P_N^{eff} = \frac{(\gamma p)^{2^N}}{\gamma}. \quad (5)$$

If the physical error probability is above $1/\gamma$, the effective error probability will increase as more redundancy is added. On the contrary, if $p \leq 1/\gamma$, there is a doubly exponential decrease of the effective error probability in the number of concatenations. For a code $[n, 1, 3]$, the number of physical qubits $Q_N = n^N$ grows exponentially in the number of concatenations, so we still have a exponential decrease of P_N^{eff} in the number of physical qubits.

The fact that we can locate loss error leads to a slightly different way of calculating the threshold, which is explained in next section.

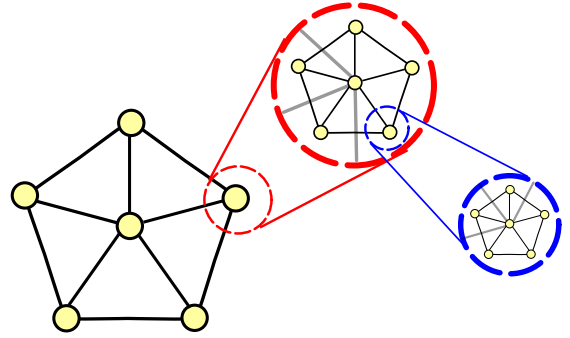


Fig. 3 We will stop at some level of concatenation that meets our loss tolerance requirements.

3 Tolerance to Loss

A general result in classical and quantum information theory [11] is that a code that protects against t errors, can protect against $2t$ losses (loss can be regarded a localized error). It is important to stress that in our scheme we only allow for destructive measurements, that is, the qubit being measured will no longer be available to extract more information. Ultimately this is the reason why tolerance to Pauli errors cannot be integrated within this approach.

We classify loss into two broad categories: preannounced and non-preannounced. Preannounced loss happens when the absence of a qubit is detected in advance of aiming to measure it, whether the system is there or not. Non-preannounced loss means that one discovers the loss only upon measuring and not getting any “click”. This categorization is not exactly the same as

the “heralded-unheralded” division, since heralded loss means that we detect a loss at measurement and tag the location, whereas unheralded means that there is a loss that is not detected. To our understanding, this new categorization fits better in the measurement-based approach to quantum computation.

To see that the logical operators have support on only three physical qubits, let us multiply them by the relevant stabilizers, as follows:

$$\bar{X} \equiv \begin{matrix} & Z_1 & & Z_1 & & I_1 \\ X_5 & I_l & I_2 & \cdot & I_5 & I_l & Y_2 \\ & Z_4 & & I_3 & & Z_4 & & Y_3 & & I_4 & & Y_3 \end{matrix} = X_5 I_l Y_2. \quad (6)$$

For \bar{Z} , we have:

$$\bar{Z} \equiv \begin{matrix} & Z_1 & & Z_1 & & I_1 \\ Z_5 & I_l & Z_2 & \cdot & I_5 & I_l & Y_2 \\ & Z_4 & & Z_3 & & Z_4 & & Y_3 & & I_4 & & X_3 \end{matrix} = Z_5 I_l X_2, \quad (7)$$

$$\bar{Y} \equiv \begin{matrix} & Z_1 & & Y_1 & & Z_1 & & Y_1 \\ Z_5 & I_l & Z_2 & \cdot & Y_5 & I_l & Z_2 & \cdot & Y_5 & I_l & I_2 \\ & Z_4 & & Z_3 & & Z_4 & & I_3 & & Y_4 & & Z_3 & & Y_4 & & I_3 \end{matrix} = Z_5 I_l I_2. \quad (8)$$

and all their variations derived from rotational symmetry. Thus we only need to measure three out of five qubits to retrieve the logical information. Similarly one can measure the \bar{Y} operator.

3.1 Results for Preannounced Loss

In the case where we have knowledge about the location of lost qubits, a threshold for the loss probability can be derived which coincides with the theoretical maximum of 50%. If this maximum could be surpassed, then we would be able to copy quantum information in arbitrary basis which, as we saw in the Introduction, is precluded by the no-cloning theorem [17].

We consider a physical qubit loss probability p_L . We show that under concatenation the effective loss probability decreases exponentially in the number of concatenations. The recurrence formula

$$P_L(N-1) = \binom{5}{5} P_L^5(N) + \binom{5}{4} P_L^4(N)(1 - P_L(N)) + \binom{5}{3} P_L^3(N)(1 - P_L(N))^2 \quad (9)$$

gives us the effective loss probability at concatenation level $N-1$, $P_L(N-1)$, given that the loss probability at level N was $P_L(N)$, with $P_L(N_{top}) = p_L$ at the top level. By plotting $P_L(N)$ for different N it is possible

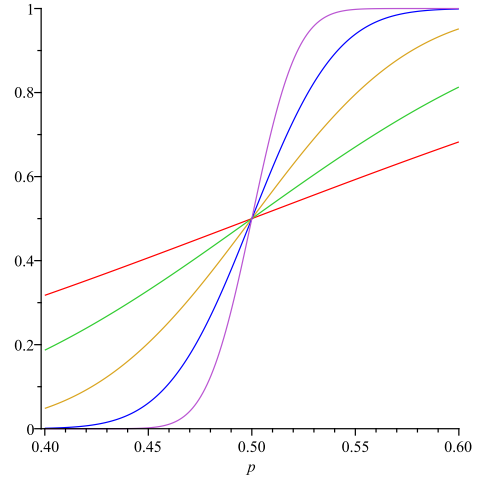


Fig. 4 Effective loss probability versus physical loss probability for preannounced loss. Different levels of concatenation, from $N = 1$ (cf. $Q = 5$), red, to $N = 5$ (cf. $Q = 3125$), purple.

come up with a recurrence fixed point corresponding to a threshold of 50%, as given in Fig. 4.

The recurrence formula giving the effective loss probability assumes that only the top level of concatenation is actually exposed to loss. This means that we should regard all the previous concatenation levels as levels of virtual qubits that help us visualize how to construct the code. These virtual levels are also useful in order to visualize the decoding procedure. Each virtual qubit in level $N-1$ is encoded in five qubits in level N , and N_{top} corresponds to the actual physical level. Being unable to recover the information stored in any pentagon belonging to level N will result in declaration of loss of the corresponding underlying qubit.

In order to gain some insight on the amount of protection given by this way of encoding for preannounced loss, Table 3 illustrates the number of qubits needed in to make the effective loss probability $P_L(N) \approx 10^{-8}$ or below.

	$p_L = 0.2$	$p_L = 0.3$	$p_L = 0.4$
Q^V	22188	2.3×10^5	7.6×10^6
Q	125	625	3125

Table 3 Number of physical qubits Q^V (for the trees approach) and Q (for this proposal) used to achieve an effective loss probability $P_L(N) \approx 10^{-8}$ or below.

where we have compared our results with the amount of qubits needed in [1]. We stress that in our case, these values are valid only when loss is preannounced, as opposed to [1] where loss need not be known beforehand. We nevertheless include this table to show that resources would be dramatically reduced if one could

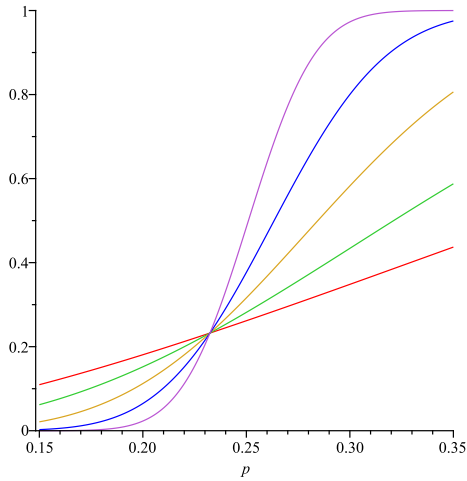


Fig. 5 Effective loss probability versus physical loss probability for non-preannounced loss. Different levels of concatenation, from $N = 1$ (cf. $Q = 5$), red, to $N = 5$ (cf. $Q = 3125$), purple.

tag lost qubits prior to measurement, such as may be relevant for atoms in optical lattices.

3.2 Results for Non-preannounced Loss

The bound of 50% is achieved when one knows whether the qubit is there or not. If one discovers a loss while performing a computational measurement, there is a chance that the measurement pattern chosen prior to discovering the loss will be unavoidably broken and the information lost. If this happens, then one declares a loss, which can be handled in the same way as loss in the immediately lower concatenation level.

Since each logical operator can be written in two different (commuting) ways, one can try to measure both ways at the same time and declare a loss whenever either two losses break both of them, or when a loss breaks one of them but it is impossible go back and to measure the other one. A decision tree will give us the probability of this happening. The threshold is considerably lowered to about 23% percent, which is, however, comparable to thresholds obtained for other architectures [4, 18].

3.2.1 Measurement strategy for non-preannounced loss

The decision tree for a \bar{Z} measurement is given in the following pseudocode. The notation “ jA ” means “qubit j is measured in the A basis”.

The algorithm reduces to an attempt to measure a correlation conforming a logical operator. A logical operator can be written in different forms, as shown for example in equations 7 and 8. Upon discovering a

loss error, the operator being measured may or may not be retrieved, depending on whether changing the basis in which the remaining qubits are measured allows to switch from one correlation to another.

Decision Tree 1 for a \bar{Z} measurement

```

1: if 1X then
2:   if 2Z then
3:     if 5Z then
4:       SUCCESS
5:     else if 3Y then
6:       if 5Y then
7:         SUCCESS
8:       else
9:         FAILURE
10:      end if
11:    else
12:      FAILURE
13:    end if
14:  else if 3Y then
15:    if 5Y then
16:      SUCCESS
17:    else
18:      FAILURE
19:    end if
20:  else
21:    FAILURE
22:  end if
23: else if 2X then
24:   if 4Y then
25:     if 5Y then
26:       SUCCESS
27:     else
28:       FAILURE
29:     end if
30:   else
31:     FAILURE
32:   end if
33: else if 4X then
34:   if 3Z then
35:     if 5Z then
36:       SUCCESS
37:     else
38:       FAILURE
39:     end if
40:   else
41:     FAILURE
42:   end if
43: else
44:   FAILURE
45: end if

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There are similar decision trees for \bar{X} and \bar{Y} measurements. Using these pseudocodes recursively, *i.e.* at each level of concatenation, will give us a measurement strategy in the limit of many concatenations. Adding probabilities for success and failure will give us the curve in Fig. 5.

4 Loss-tolerant Universal Quantum Computation

To see how universality is achieved, keep in mind that a logical graph state underlies all encodings. We assume this graph state is two-dimensional, so that two qubit logical gates (C_Z) are naturally embedded in the encoding. As shown in Fig. 2, encoding can be seen as entangling the operators of the virtual qubit with the logical operators living in the pentagons. Imagine we have two such virtual qubits entangled with a C_Z gate. It is trivial to see how this translates into entanglement between the logical operators of their respective encodings.

$$\begin{array}{ccc} \bar{I} & \bar{I} & \bar{I} & \bar{X} \\ I & Z & X & I \\ I & X & Z & I \\ \bar{X} & \bar{I} & \bar{I} & \bar{I} \end{array} \Rightarrow \begin{array}{ccc} \bar{I} & \bar{I} & \bar{Z} & \bar{X} \\ I & Z & X & Z \\ Z & X & Z & I \\ \bar{X} & \bar{Z} & \bar{I} & \bar{I} \end{array} \Rightarrow \begin{array}{ccc} \bar{X} & \bar{I} & \bar{I} & \bar{Z} \\ I & I & X & X \\ X & X & I & I \\ \bar{Z} & \bar{I} & \bar{I} & \bar{X} \end{array}, \quad (10)$$

where we start off with two virtual qubits (centre) in an entangled state. The first arrow represents the encoding of the virtual qubits (*i.e.* entangle them with logical operators), and the second arrow represents measurement of the virtual qubits in the X basis. This shows that the logical operators are entangled via a logical C_Z gate.

Unfortunately, only measurements in the X and Z basis can be done in a loss-tolerant fashion. This prevents us from doing single qubit gates with the usual prescription (*i.e.* steer a unitary by measuring in angles given by it's Euler decomposition). This can be overcome by introducing the additional set of gates $\exp(i\frac{1}{8})Z$, $\exp(i\frac{1}{4})Z$ and $\exp(i\frac{1}{4})X$. These gates can be realized fault-tolerantly using a special type of error-free states known as *magic states* [19]. Magic states can be distilled using from a reservoir of not-too-noisy ancillas using only measurements in the X and C_Z gates.

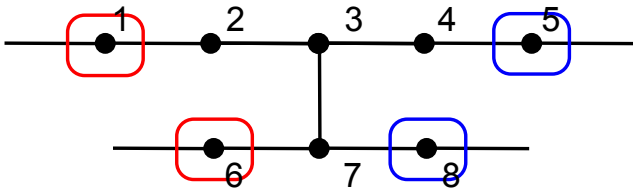


Fig. 6 This is how a C_X gate would look like in the virtual graph state. Red squares denote input states and blue squares denote output states. All qubits except the blue ones are measured in the X basis.

The correlations defining the C_X gate of Fig. 6 are:

$$X_1 I_2 X_3 I_4 X_5 I_7 X_8, \quad (11)$$

$$Z_1 X_2 I_3 X_4 Z_5, \quad (12)$$

$$I_3 X_4 Z_5 Z_6 X_7 Z_8, \quad (13)$$

$$X_6 I_7 X_8. \quad (14)$$

Measuring all qubits of Fig. 6 in the X basis will enact a C_X gate. Hadamard gates are also straightforward to achieve. This is clear if we again have a look at the virtual graph state, since measuring in the X basis a qubit in state $|\psi\rangle$ will steer the next qubit in the graph into the state $X^m H |\psi\rangle$, where m is the outcome of the measurement.

4.1 Overhead

The basic principle of fault tolerance using concatenated codes is that, whenever the physical qubit error probability is below the threshold, the effective error probability decreases exponentially with the number of physical qubits. However, as one gets closer to the threshold, the resources needed to maintain a given effective loss probability P_L increase very fast. The effective loss probability P_L as a function of the overhead $Q = 5^N$ and the physical loss probability p_L is summarized in Tables 4 and 5 for preannounced and non-preannounced loss, respectively.

Q_P	$p_L = 0.4$	$p_L = 0.3$	$p_L = 0.2$
5	0.317	0.163	0.058
25	0.187	0.033	0.002
125	0.048	3.6×10^{-4}	5.6×10^{-8}
625	0.001	4.5×10^{-10}	1.8×10^{-21}
3125	1.5×10^{-8}	9.1×10^{-28}	5.5×10^{-62}

Table 4 Effective loss probability as a function of the number of qubits Q_P and the physical loss probability p_L , for the case of preannounced loss.

Q_{NP}	$p_L = 0.15$	$p_L = 0.1$	$p_L = 0.05$
5	0.110	0.052	0.014
25	0.062	0.015	0.001
125	0.021	0.001	8.0×10^{-6}
625	0.002	1.1×10^{-5}	3.8×10^{-10}
3125	4.1×10^{-5}	7.7×10^{-10}	8.9×10^{-19}

Table 5 Effective loss probability as a function of the number of qubits Q_{NP} and the physical loss probability p_L , for the case of non-preannounced loss.

4.2 Comparison with Tree Codes

We show now that, for low enough p_L , this approach necessitates less resources than a previous scheme which attains also the highest threshold achievable. It was introduced by M. Varnava et al. in [1] and offers protection to loss for p_L up to 50%. It is remarkable that this way of encoding a qubit protects against non-preannounced loss. The graph states they introduce consists of “trees” of qubits were at each level of the tree structure, the branching parameter is potentially different from the others. Results were obtained by exhaustive search over the branching parameter.

We investigated whether there is a regime in which the approach of concatenated pentagons performs better, in terms of overhead needed, than the tree approach. We found that this is the case for a non-preannounced loss probability below $\sim 10\%$, as can be inferred from Fig. 7.

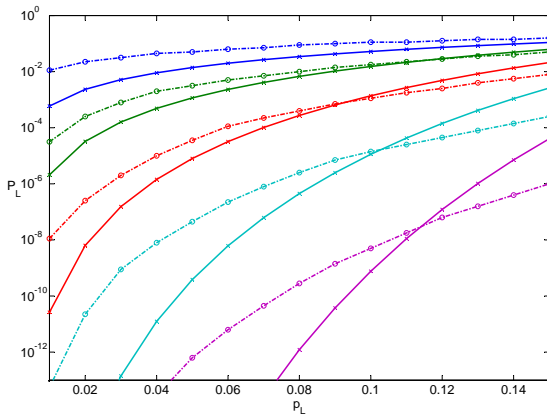


Fig. 7 Comparison of the effective loss probability P_L as a function of the physical loss probability p_L , for both the current approach (solid lines) and the tree codes (dashed lines). The colors correspond to different numbers of redundant physical qubits Q . Blue lines correspond to $Q \approx 5$ qubits, green to $Q \approx 25$, red to $Q \approx 125$, cyan to $Q \approx 625$ and purple to $Q \approx 3125$. Clearly, the current approach yields better protection for low loss probabilities.

To summarize, this scheme requires to our knowledge the least overhead for preannounced loss. For non-preannounced loss, it performs better than the trees below $\sim 10\%$, which is to date the scheme with the best performance for non-preannounced loss.

5 Conclusions and Outlook

We have introduced a new way to fight loss errors and shown that it allows for universal quantum computa-

tion. Comparing this with Varnava’s results, we see that this new approach demands significantly less overhead for loss rates below $\sim 10\%$. For preannounced loss, this scheme saturates the upper bound given by the no-cloning theorem with very few resources.

However, it seems difficult to combine it with tolerance to computational errors since $\Omega(5^N)$ gates are needed in order to provide protection at N levels of concatenation. This, even for very low unknown error probabilities, will effectively randomize the encoded qubit, since it only takes one wrong measurement to change the parity of the logical qubit. There are, however, architectures such as optical quantum computing, where qubits are in an essentially zero temperature environment, so loss is a far more significant error than Pauli errors.

We analyzed other codes, such as the four qubit code and the seven qubit code, but their performance was significantly worse than the five qubit code, so we didn’t continue in that direction. An interesting open question is whether larger rings of qubits can give rise to the same loss protection while increasing the number of encoded qubits.

Also, there is an exciting connection between the erasure channel and access structures [20]. The concatenated nature of this scheme gives rise to a fractal network topology, which potentially arises in several networking scenarios. A more in depth study of this connection would thus be interesting.

6 Acknowledgments

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